KEK-TH-449
hep-th/9509025
September 1995

Ward Identities of W_∞ Symmetry and Higher Genus Amplitudes in 2D String Theory

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(*Revised version published in Nucl. Phys. B*)

Abstract

The Ward identities of the W_∞ symmetry in two dimensional string theory in the tachyon background are studied in the continuum approach. We consider amplitudes different from 2D string ones by the external leg factor and derive the recursion relations among them. The recursion relations have non-linear terms which give relations among the amplitudes defined on different genus. The solutions agree with the matrix model results even in higher genus. We also discuss differences of roles of the external leg factor between the $c_M = 1$ model and the $c_M < 1$ model.

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1 Introduction and Summary

Many interesting issues in string theory such as dynamically compactification, black hole physics, etc, require a non-perturbative formulation. Such a formulation is not now available in higher dimensional string theories. In two or fewer spacetime dimensions, however, the string theory becomes solvable [1] due to the presence of an infinite number of W_∞ currents [2, 3, 4, 5, 6, 7, 8], which gives a possibility of studying the non-perturbative formulation of string theory. Furthermore 2D string theory itself is interesting in spacetime physics. It gives the 2D dilaton gravity with a massless matter called "tachyon" as the effective theory [9]. Thus 2D string theory is also attractive as an alternative approach to studying 2D quantum dilaton gravity [10].

There are several formulations of two dimensional string theory. The matrix model (see reviews [1]) is generally believed to describe the 2D string theory, which is in principle defined non-perturbatively. The continuum theory [11, 12, 13] is defined using the standard quantization method of the string perturbation theory. The topological description [15] of 2D string theory is formulated in [16, 7]. To understand the non-perturbative formulation of string theory it is important to clarify the relations between these methods. In the present work we investigate the continuum method of 2D string theory. We consider the Ward identities of the W_∞ symmetry for amplitudes in the tachyon background and give the recursion relations which connect amplitudes on different genus. We then study the correspondence between the matrix model and the continuum theory genus by genus.

String theory in 2D target space-time is described by a combination of the Liouville and the $c_M = 1$ matter fields. The Liouville and the matter fields are identified with the space and the time coordinates in the target space. In the next section we define the S-matrix of the massless tachyons in the tachyon background. Here we introduce the $\hat{S}$-matrix, which is different from the S-matrix by the external leg factor. We will see that the $\hat{S}$-matrix is equivalent to the amplitude of the $c_M = 1$ matrix model even in higher genus. In Sect.3 we set up the Ward identities of the W_∞ symmetry [2]. We

[2] The Ward identities of W_∞ symmetry for $c_M = 1$ theory were first discussed in ref. [3]. They considered the *linear* Ward identities, which, however, do not give correct answers except for special cases of amplitudes. In general we need the non-linear terms as discussed in this paper.

give the boundary formulas contributing to the identities, which are obtained by taking the $c_M \to 1$ limit of the results computed in the previous work for the $c_M < 1$ model [4]. In Sect.4 we write out several Ward identities using the contributions calculated in Sect.3 and solve them. We first consider sphere amplitudes. In this case we can give the procedure to derive all types of sphere amplitudes. We first give the derivation of the $1 \to N$ amplitudes and then the $2 \to N$ ones. Using this ordering we can recursively obtain all further types of sphere amplitudes. We here explicitely calculate the $1 \to N$, the $2 \to 2$ and the $2 \to 3$ amplitudes for all kinematical regions, which agree with the results calculated in the matrix model [1] and also the topological 2D string [7]. For higher genus cases we need further discussions. For the current $W_{-n,-m}$ with $n, m \leq 2$, the identities are closed within the contributions calculated in Sect.3. But, for the general W_∞ currents, extra boundary contributions are needed. Their direct calculations are very difficult. Here, we guess the formula from a simple argument and check that the identities are indeed closed. We also give the procedure to calculate higher genus amplitudes recurrently. Thus we confirm that the solutions of the identities agree with the matrix model results [1, 18] up to genus three. In the last part of this section we also calculate the partition function of 2D string theory. In Sect.5 we discuss differences of structures, especially roles of the external leg factor, between the $c_M = 1$ model and the $c_M < 1$ model.

2 Scattering Amplitudes of Tachyons

Two dimensional string theory is defined by the action [3]

$$I_0 = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{g}} (\hat{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \hat{g}^{\alpha\beta} \partial_\alpha X \partial_\beta X + 2\hat{R}\phi) \ , \tag{2.1}$$

where ϕ is the Liouville field and X is the $c_M = 1$ matter field. In spacetime physics ϕ is identified with the space coordinate and X is the (Euclidean) time.

A physical state with continuous momentum can only be the massless scalar called "tachyon" in the string terminology. In two dimensions the tachyon mode becomes massless. Besides that there are an infinite number

[3] Note that the normalization of the fields differs from that in ref. [4] by $\sqrt{2}$.

of the physical states that exist only at discrete momenta, called the discrete states [17]. They are discussed in the next section.

The tachyon vertex operator with momentum k (>0) is given by

$$T_k^{\pm} = \frac{1}{\pi}\int d^2z e^{(2-k)\phi(z,\bar{z})\pm ikX(z,\bar{z})} , \tag{2.2}$$

where $\pm$ denotes the chirality. The selection of $k>0$ is called the Seiberg condition [12]. We will postulate below that the amplitude including the anti-Seiberg $(k<0)$ states vanishes. Henceforth we introduce the normalized operator

$$\hat{T}_k^{\pm} = \Lambda(k) T_k^{\pm} , \qquad \Lambda(k) = \frac{\Gamma(k)}{\Gamma(-k)} . \tag{2.3}$$

Let us consider the action in the tachyon background

$$I = I_0 + \mu_B T_0 , \tag{2.4}$$

where $T_0 = \lim_{\epsilon\to 0}\hat{T}_\epsilon^{\pm}$. The bare tachyon background μ_B is divergent, which is renormalized as follows:

$$\mu_B\, \epsilon \rightarrow \mu . \tag{2.5}$$

This ensures the non-decoupling of $\mu_B T_0$.

The S-matrix of tachyons in the tachyon background is defined by

$$S^{(g)}_{k_1,\cdots,k_N\to p_1,\cdots,p_M} = \prod_{i=1}^{N}\Lambda^{-1}(k_i)\prod_{j=1}^{M}\Lambda^{-1}(p_j)\ \hat{S}^{(g)}_{k_1,\cdots,k_N\to p_1,\cdots,p_M} , \tag{2.6}$$

where the $\hat{S}$-matrix is

$$\begin{aligned}\hat{S}^{(g)}_{k_1,\cdots,k_N\to p_1,\cdots,p_M} &=< \prod_{i=1}^{N}\hat{T}^{+}_{k_i}\prod_{j=1}^{M}\hat{T}^{-}_{p_j} >_g \\ &= \left(-\frac{\lambda}{2}\right)^{-\chi/2}\delta\Big(\sum_{i=1}^{N}k_i-\sum_{j=1}^{M}p_j\Big)\ \mu_B^s\frac{\Gamma(-s)}{2} < \prod_{i=1}^{N}\hat{T}^{+}_{k_i}\prod_{j=1}^{M}\hat{T}^{-}_{p_j}\ (T_0)^s >_g^{(free)} ,\end{aligned} \tag{2.7}$$

The superscript $free$ denotes the free field representation. The δ-function and $\mu_B^s\frac{\Gamma(-s)}{2}$ come from the zero-mode integrals of X and ϕ respectively [14]. g is the genus, $\chi = 2-2g$ and s is given by

$$s = \sum_{i=1}^{N}k_i + \chi - N - M . \tag{2.8}$$

Thus we introduce an $\hat{S}$-matrix different from the S-matrix of two dimensional string theory by the external leg factor Λ^{-1}. We will see that the $\hat{S}$-matrix is equivalent to the amplitudes of the $c_M = 1$ matrix model even in higher genus.

Finally we note that the $\hat{S}$-matrix is invariant under the interchange of chiralities,

$$< \prod_{i=1}^{N} \hat{T}^{+}_{k_i} \prod_{j=1}^{M} \hat{T}^{-}_{p_j} >_g = < \prod_{i=1}^{N} \hat{T}^{-}_{k_i} \prod_{j=1}^{M} \hat{T}^{+}_{p_j} >_g \tag{2.9}$$

or $\hat{S}^{(g)}_{k_1,\cdots,k_N \to p_1,\cdots,p_M} = \hat{S}^{(g)}_{p_1,\cdots,p_M \to k_1,\cdots,k_N}$.

3 Ward Identities of W_∞ symmetry

There is an infinite number of BRST invariant states at discrete momenta called the discrete states [17]. Here we consider the discrete states of ghost number zero, $B_{r,\ s}$ and one, $\Psi_{r,\ s}$ with the parametrization by two negative integers r and s, which have the Liouville and the matter momenta $\alpha_{r,\ s} = 2 + r + s$ and $\beta_{r,\ s} = r - s$. The states $\Psi_{r,\ s}$ are the remnants of the massive string modes, which are constructed from the OPE of the tachyon operators with special momenta,

$$V^{-}_{-r+1}(z,\bar{z})V^{+}_{-s+1}(w,\bar{w}) \sim \frac{1}{|z-w|^2} R_{r,\ s}(w)\bar{R}_{r,\ s}(\bar{w}) \ , \tag{3.1}$$

where $V^{\pm}(z,\bar{z})$ is the exponential part of the tachyon operator (2.2) and $R_{r,\ s}(z) = (b_{-1}\Psi_{r,\ s})(z)$.

The W_∞ symmetry currents are constructed from these states as is shown below [2, 3, 4]. The discrete states $R_{r,\ s}$ ($r,\ s \in \mathbf{Z}_-$) form the W_∞ algebra. Here we normalize the fields such that

$$R_{r,\ s}(z)R_{r',\ s'}(w) = \frac{1}{z-w}(rs' - r's)R_{r+r'+1,\ s+s'+1}(w) \ . \tag{3.2}$$

The ghost number zero states $B_{r,\ s}$ have the ring structure

$$B_{r,\ s}(z)B_{r',\ s'}(w) = B_{r+r'+1,\ s+s'+1}(w) \ . \tag{3.3}$$

Combining $R_{r,\ s}(z)$ and $\bar{B}_{r,\ s}(\bar{z})$, we can construct the symmetry currents

$$W_{r,\ s}(z,\bar{z}) = R_{r,\ s}(z)\bar{B}_{r,\ s}(\bar{z}) \ , \tag{3.4}$$

which satisfy

$$\partial_{\bar z} W_{r,\ s}(z,\bar z) = \{\bar Q_{BRST}, [\bar b_{-1}, W_{r,\ s}(z,\bar z)]\}\ . \tag{3.5}$$

In the following we consider the Ward identities of the currents

$$\frac{1}{\pi}\int d^2z\partial_{\bar z} < W_{r,\ s}(z,\bar z)\ \mathcal{O} >_g = 0\ , \qquad r,s\in \mathbf{Z}_-\ , \tag{3.6}$$

where $\mathcal{O}$ is a product of the normalized tachyon operators (2.3).

Let us calculate the operator product expansions (OPE) between the currents and the tachyon operators. They are given by taking the $c_M \to 1$ limit of the previous work for $c_M < 1$ [4]. For $c_M < 1$ theory the tachyon operator $\hat O_j$, which is identified with the gravitationally dressed scaling operator, has discrete momentum parametrized by the positive integer j. Taking the limit $p,\ q \to \infty$ ($p - q =$ finite) [4] and $j/q \to k$ such that $\frac{1}{\pi}\hat O_j \to \hat T_k^+$, we then obtain

$$\begin{aligned} &W_{-n,-m}(z,\bar z)\ \hat T^+_{k_1}(0,0)\ \hat T^+_{k_2}\cdots \hat T^+_{k_n} \\ &\quad = \frac{1}{z}\ n!\ \left(\prod_{i=1}^{n} k_i\right)\ \hat T^+_{k_1+\cdots+k_n-n+m}(0,0)\ , \end{aligned} \tag{3.7}$$

where $\hat T_k^+(z,\bar z)$ is defined by replacing the integral in (2.2) with $\bar c(\bar z)c(z)$. This contribution is graphically expressed in fig.1. Note that the OPE with the zero momentum tachyon T_0 vanishes, but the OPE with the tachyon background $\mu_B T_0$ becomes finite due to the renormalization (2.5).

The OPE with the tachyon $\hat T_p^-$ is easily calculated by changing the chirality. It is carried out by changing the sign of the field X such that $\hat T^+ \to \hat T^-$ and $W_{r,\ s} \to -W_{s,\ r}$ [5]. We then get

$$\begin{aligned} &W_{-n,-m}(z,\bar z)\ \hat T^-_{p_1}(0,0)\ \hat T^-_{p_2}\cdots \hat T^-_{p_m} \\ &\quad = -\frac{1}{z}\ m!\ \left(\prod_{j=1}^{m} p_j\right)\ \hat T^-_{p_1+\cdots+p_m+n-m}(0,0)\ . \end{aligned} \tag{3.8}$$

The OPE's (3.7) and (3.8) have already been computed in [3].

[4] Here $c_M = 1 - 6(p-q)^2/pq$.

[5] In detail $R_{r,\ s} \to -R_{s,\ r}$ and $B_{r,\ s} \to B_{s,\ r}$. These changes preserve the OPE's (3.2) and (3.3) respectively.

The derivative $\partial_{\bar z}$ in (3.6) picks up the OPE singularities (3.7) and (3.8), which give the linear terms of the Ward identities. In addition we get the correlator

$$< \frac{1}{\pi} \int d^2 z \{ \bar{Q}_{BRST}, [\bar{b}_{-1}, W_{r,\ s}(z, \bar{z})] \} \ \mathcal{O} >_g \quad . \tag{3.9}$$

This correlator does not vanish, which gives the anomalous contributions from the boundary of moduli space as in the case of $c_M < 1$ theory. The boundary is described by using the propagator in the form

$$D = \frac{1}{\pi} \int_{\mathrm{e}^{-\tau} \le |z| \le 1} \frac{d^2 z}{|z|^2} z^{L_0} \bar{z}^{\bar{L}_0} = 2 \Big(\frac{1}{H} - \frac{1}{H} \mathrm{e}^{-\tau H} \Big) \ , \tag{3.10}$$

where $H = L_0 + \bar{L}_0$ and L_0 is the zero mode of the Virasoro generator including the ghost part: $L_0 = L_0^L + L_0^M + L_0^G$. The last term, at $\tau \to \infty$, stands for the boundary of the moduli space. The anomalous contributions are also calculated by taking the limit $p,\ q \to \infty$ ($p - q =$ finite) and then by replacing the summation of the discrete tachyon momentum with the integral of the continuous one such as, for example, $\frac{\pi}{q} \sum_{j=1}^{nq-1} \to \int_0^n dl$ in ref. [4].

There are many types of anomalous contributions. The boundary configuration that gives the product of two amplitudes is shown in fig.2. At $\tau \to \infty$ it gives the following contribution for $(r, s) = (-n, -m)$:

$$\lambda \ n! \left(\prod_{i=1}^{n-1} k_i \right) \int_0^{\sum_{i=1}^{n-1} k_i - n + m} dl < \mathcal{O}_1' \ \hat{T}^+_{\sum_{i=1}^{n-1} k_i - n + m - l} >_{g_1} < \hat{T}_l^+ \ \mathcal{O}_2' >_{g_2} \ , \tag{3.11}$$

where $g_1 + g_2 = g$. The primes on $\mathcal{O}_1$ and $\mathcal{O}_2$ stand for the exclusion of the operators $\hat{T}^+_{k_i}$ $(i = 1, \cdots, n-1)$, where $\hat{T}^+_{k_i}$'s are the operators in $\mathcal{O}$ or the tachyon background $-\mu_B T_0$. The partition of the set $\mathcal{O}$ into $\mathcal{O}_1'$ and $\mathcal{O}_2'$ is done in such a way that there is no overcounting. The partition of genus g into g_1 and g_2 is also done.

There is a variant of the contribution (3.11). The boundary configuration that the surfaces of g_1 and g_2 are connected by a handle (see fig.3) gives the following contribution:

$$\lambda \ n! \left(\prod_{i=1}^{n-1} k_i \right) \int_0^{\sum_{i=1}^{n-1} k_i - n + m} dl \frac{1}{2!} < \hat{T}^+_{\sum_{i=1}^{n-1} k_i - n + m - l} \hat{T}_l^+ \ \mathcal{O}' >_{g-1} \ , \tag{3.12}$$

where the factor $1/2!$ corrects the double counting coming from the interchange of $\hat{T}^+_{\sum_{i=1}^{n-1} k_i - n + m - l}$ and $\hat{T}_l^+$.

The boundary contribution with the triple product of amplitudes is also calculated by using the result in ref.[4] in the form

$$\lambda^2 \; n! \left(\prod_{i=1}^{n-2} k_i\right) \int_0^{\sum_{i=1}^{n-2} k_i - n + m} dl \int_0^{\sum_{i=1}^{n-2} k_i - n + m - l} dl'$$
$$\times < \mathcal{O}'_1 \; \hat{T}^+_{\sum_{i=1}^{n-2} k_i - n + m - l - l'} >_{g_1} < \hat{T}^+_l \; \mathcal{O}'_2 >_{g_2} < \hat{T}^+_{l'} \; \mathcal{O}'_3 >_{g_3} \quad (3.13)$$

where $g_1 + g_2 + g_3 = g$. There are some variants of the contributions (3.13). For instance, from the configuration that the surfaces of g_1 and g_2 are connected by a handle, we get the contribution given by carrying out the following replacement for the formula (3.13):

$$< \mathcal{O}'_1 \; \hat{T}^+_{\sum_{i=1}^{n-2} k_i - n + m - l - l'} >_{g_1} < \hat{T}^+_l \; \mathcal{O}'_2 >_{g_2}$$
$$\to \frac{1}{2!} < \hat{T}^+_{\sum_{i=1}^{n-2} k_i - n + m - l - l'} \hat{T}^+_l \; \mathcal{O}'_{1+2} >_{g_1+g_2-1} \quad . \quad (3.14)$$

In the case that the three surfaces are connected by handles with each other, we then get the contribution given by replacing the triple product term with

$$\frac{1}{3!} < \hat{T}^+_{\sum_{i=1}^{n-2} k_i - n + m - l - l'} \hat{T}^+_l \hat{T}^+_{l'} \; \mathcal{O}' >_{g-2} \quad . \quad (3.15)$$

In general the contributions are expressed as follows (see fig.4):

$$\lambda^{a-1} \; n! \left(\prod_{i=1}^{n+1-a} k_i\right) \int \prod_{i=1}^{a} dl_i \theta(l_i) \; \delta\Big(\sum_{i=1}^{a} l_i - \sum_{i=1}^{n+1-a} k_i + n - m\Big)$$
$$\times < \hat{T}^+_{l_1} \; \mathcal{O}'_1 >_{g_1} < \hat{T}^+_{l_2} \; \mathcal{O}'_2 >_{g_2} \cdots < \hat{T}^+_{l_a} \; \mathcal{O}'_a >_{g_a} \quad , \quad (3.16)$$

where $\sum_{i=1}^{a} g_i = g$ and $a = 1, \cdots, n+1$. θ is the step function. The $a = 1$ formula is nothing but the contribution of the OPE (3.7). In addition, as discussed in the cases of $a = 2$ and 3, there are many variants of this expression, which come from the boundary configurations that some of the surfaces are connected by handles.

For the negative chirality we get

$$-\lambda^{b-1} \; m! \left(\prod_{j=1}^{m+1-b} p_j\right) \int \prod_{i=1}^{b} dl_i \theta(l_i) \; \delta\Big(\sum_{i=1}^{b} l_i - \sum_{j=1}^{m+1-b} p_j - n + m\Big)$$
$$\times < \hat{T}^-_{l_1} \; \mathcal{O}'_1 >_{g_1} < \hat{T}^-_{l_2} \; \mathcal{O}'_2 >_{g_2} \cdots < \hat{T}^-_{l_b} \; \mathcal{O}'_b >_{g_b} \quad , \quad (3.17)$$

where $\sum_{i=1}^{b} g_i = g$ and $b = 1, \cdots, m+1$. $b = 1$ corresponds to the contribution of the OPE (3.8). And also there are many variants of these contributions as mentioned above.

4 Recursion Relations

In this section we write out several recursion relations and discuss the structures of them. We consider the recursion relations for sphere amplitudes at first and then discuss the case of higher genus ones. For the higher genus cases we need to add the extra boundaries not discussed in Sect.3.

4.1 Sphere amplitudes

Let us first consider the $W_{-n,-1}$ identity of type

$$\frac{1}{\pi}\int \bar{\partial} < W_{-n,-1}\, \hat{T}^{+}_{k_1} \prod_{j=1}^{M} \hat{T}^{-}_{p_j} >_0 = 0 \ . \tag{4.1}$$

For the first few values of M, these equations become as follows. $M = 1$ gives the linear identity

$$\begin{aligned} -x < \hat{T}^{+}_{k_1}\hat{T}^{-}_{p_1}\hat{T}^{-}_{n-1} >_0 -p_1 < \hat{T}^{+}_{k_1}\hat{T}^{-}_{p_1+n-1} >_0 & \\ +nx^{n-1}k_1 < \hat{T}^{+}_{k_1-n+1}\hat{T}^{-}_{p_1} >_0 = 0 \ . & \end{aligned} \tag{4.2}$$

where $x = -\mu$. The first and the second terms of the l.h.s. come from the OPE (3.8) with the background $-\mu_B T_0$ $(= \lim_{p\to 0}(x/p)\ \hat{T}^{-}_{p})$ and the tachyon $\hat{T}^{-}_{p_1}$ respectively. The third term comes from the OPE (3.7) with $\hat{T}^{+}_{k_1}$ and $n-1$ $-\mu_B T_0$'s $(= \lim_{k\to 0}(x/k)\ \hat{T}^{+}_{k})$, where $n = n!/(n-1)!$ and $(n-1)!$ denotes the permutation of $-\mu_B T_0$'s. For $M = 2$ the identity becomes

$$\begin{aligned} & -x < \hat{T}^{+}_{k_1}\hat{T}^{-}_{p_1}\hat{T}^{-}_{p_2}\hat{T}^{-}_{n-1} >_0 -p_1 < \hat{T}^{+}_{k_1}\hat{T}^{-}_{p_1+n-1}\hat{T}^{-}_{p_2} >_0 \\ & -p_2 < \hat{T}^{+}_{k_1}\hat{T}^{-}_{p_1}\hat{T}^{-}_{p_2+n-1} >_0 +nx^{n-1}k_1 < \hat{T}^{+}_{k_1-n+1}\hat{T}^{-}_{p_1}\hat{T}^{-}_{p_2} >_0 \\ & +\lambda n(n-1)x^{n-2}k_1 \int_0^{k_1-n+1} dl < \hat{T}^{-}_{p_1}\hat{T}^{+}_{k_1-n+1-l} >_0 < \hat{T}^{+}_{l}\hat{T}^{-}_{p_2} >_0 = 0 \ . \end{aligned} \tag{4.3}$$

The last non-linear term comes from the anomalous contribution (3.11). For $M=3$ a triple term appears in the expression,

$$\begin{aligned}
&-x < \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_{p_2}\hat{T}^-_{p_3}\hat{T}^-_{n-1} >_0 -p_1 < \hat{T}^+_{k_1}\hat{T}^-_{p_1+n-1}\hat{T}^-_{p_2}\hat{T}^-_{p_3} >_0 \\
&-p_2 < \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_{p_2+n-1}\hat{T}^-_{p_3} >_0 -p_3 < \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_{p_2}\hat{T}^-_{p_3+n-1} >_0 \\
&+nx^{n-1}k_1 < \hat{T}^+_{k_1-n+1}\hat{T}^-_{p_1}\hat{T}^-_{p_2}\hat{T}^-_{p_3} >_0 \qquad\qquad (4.4)\\
&+\lambda n(n-1)x^{n-2}k_1 \int_0^{k_1-n+1} dl \Big[< \hat{T}^-_{p_1}\hat{T}^-_{p_2}\hat{T}^+_{k_1-n+1-l} >_0 < \hat{T}^+_l \hat{T}^-_{p_3} >_0 \\
&\qquad\qquad + < \hat{T}^-_{p_1}\hat{T}^-_{p_3}\hat{T}^+_{k_1-n+1-l} >_0 < \hat{T}^+_l \hat{T}^-_{p_2} >_0 \\
&\qquad\qquad\quad + < \hat{T}^-_{p_2}\hat{T}^-_{p_3}\hat{T}^+_{k_1-n+1-l} >_0 < \hat{T}^+_l \hat{T}^-_{p_1} >_0 \Big] \\
&+\lambda^2 n(n-1)(n-2)x^{n-3}k_1 \int_0^{k_1-n+1} dl \int_0^{k_1-n+1-l} dl' \\
&\qquad \times < \hat{T}^-_{p_1}\hat{T}^+_{k_1-n+1-l-l'} >_0 < \hat{T}^+_l \hat{T}^-_{p_2} >_0 < \hat{T}^+_{l'} \hat{T}^-_{p_3} >_0 = 0 \ .
\end{aligned}$$

In general an M-ple term appears for $M \le n$, but for $M > n$ at most an n-ple term does. Note that in this case the $(n+1)$-ple term vanishes by the Seiberg condition.

We also consider the Ward identities of type

$$\frac{1}{\pi}\int \bar{\partial} < W_{-n-1,-2}\ \hat{T}^+_{k_1} \prod_{j=1}^{M} \hat{T}^-_{p_j} >_0 = 0 \ . \tag{4.5}$$

From the $M=1$ equation we get

$$\begin{aligned}
-x^2 < \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_{n-1} >_0 -2!xp_1 < \hat{T}^+_{k_1}\hat{T}^-_{p_1+n-1} >_0 \\
+(n+1)x^n k_1 < \hat{T}^+_{k_1-n+1}\hat{T}^-_{p_1} >_0 = 0 \ .
\end{aligned} \tag{4.6}$$

After removing the amplitude $< \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_{n-1} >_0$ by using the eqs.(4.2) and (4.6), we obtain

$$-\,p_1 < \hat{T}^+_{k_1}\hat{T}^-_{p_1+n-1} >_0 + x^{n-1}k_1 < \hat{T}^+_{k_1-n+1}\hat{T}^-_{p_1} >_0 = 0 \ . \tag{4.7}$$

Solving this equation we get $< \hat{T}^+_{q_1}\hat{T}^-_{q_2} >_0 = (A/\lambda)\delta(q_1-q_2)q_1 x^{q_1+c}$, where A and c are constants. We also, using eq.(4.2) or (4.6), get $< \hat{T}^+_{q_1}\hat{T}^-_{q_2}\hat{T}^-_{q_3} >_0 =$

$(A/\lambda)\delta(q_1-q_2-q_3)q_1q_2q_3x^{q_1-1+c}$. To determine the constants A and c we further consider the $M=2$ identity of (4.5),

$$\begin{aligned}
&-x^2 < \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_{p_2}\hat{T}^-_{n-1} >_0 -2!p_1p_2 < \hat{T}^+_{k_1}\hat{T}^-_{p_1+p_2+n-1} >_0 \\
&-2!xp_1 < \hat{T}^+_{k_1}\hat{T}^-_{p_1+n-1}\hat{T}^-_{p_2} >_0 -2!xp_2 < \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_{p_2+n-1} >_0 \\
&+(n+1)x^nk_1 < \hat{T}^+_{k_1-n+1}\hat{T}^-_{p_1}\hat{T}^-_{p_2} >_0 \\
&+\lambda(n+1)nx^{n-1}k_1\int_0^{k_1-n+1} dl < \hat{T}^-_{p_1}\hat{T}^+_{k_1-n+1-l} >_0 < \hat{T}^+_l\hat{T}^-_{p_2} >_0 = 0\ .
\end{aligned} \tag{4.8}$$

Solving the simultaneous equation of (4.3) and (4.8), we obtain $A=1$, $c=0$ and the $1\to 3$ amplitude. Substituting the $1\to 1,2,3$ amplitudes into eq.(4.4) we get the $1\to 4$ amplitude.

The $1\to N-1$ amplitude is given in order by solving the Ward identity (4.1) (or (4.5)),

$$\hat{S}^{(0)}_{q_1\to q_2} = \lambda^{-1}\delta(q_1-q_2)\ q_1\ x^{q_1} \tag{4.9}$$

and

$$\hat{S}^{(0)}_{q_1\to q_2,\cdots,q_N} = \lambda^{-1}\delta\Big(q_1-\sum_{j=2}^{N}q_j\Big)\prod_{j=1}^{N}q_j\prod_{t=1}^{N-3}(q_1-t)\ x^{q_1-N+2} \tag{4.10}$$

for $N>2$. We check this formula up to $N=6$ by using the $M=4$ identity of (4.1).

Next we consider the Ward identities with two $\hat{T}^+$ tachyons,

$$\frac{1}{\pi}\int\bar{\partial} < W_{-n,-1}\ \hat{T}^+_{k_1}\hat{T}^+_{k_2}\prod_{j=1}^{M}\hat{T}^-_{p_j} >_0 = 0\ . \tag{4.11}$$

For $M=1$ we get the equation

$$\begin{aligned}
&-x < \hat{T}^+_{k_1}\hat{T}^+_{k_2}\hat{T}^-_{p_1}\hat{T}^-_{n-1} >_0 -p_1 < \hat{T}^+_{k_1}\hat{T}^+_{k_2}\hat{T}^-_{p_1+n-1} >_0 \\
&+n(n-1)x^{n-2}k_1k_2 < \hat{T}^+_{k_1+k_2-n+1}\hat{T}^-_{p_1} >_0 \\
&+nx^{n-1}\Big[k_1 < \hat{T}^+_{k_1-n+1}\hat{T}^+_{k_2}\hat{T}^-_{p_1} >_0 +k_2 < \hat{T}^+_{k_1}\hat{T}^+_{k_2-n+1}\hat{T}^-_{p_1} >_0\Big] \\
&-\lambda\int_0^{n-1} dl\Big[< \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_{n-1-l} >_0 < \hat{T}^-_l\hat{T}^+_{k_2} >_0 \\
&\qquad\qquad + < \hat{T}^+_{k_1}\hat{T}^-_{n-1-l} >_0 < \hat{T}^-_l\hat{T}^-_{p_1}\hat{T}^+_{k_2} >_0\Big] = 0\ .
\end{aligned} \tag{4.12}$$

The first term is the $2 \to 2$ amplitude. The other terms are given by the solutions (4.9-10). Thus we can get the $2 \to 2$ amplitude. To solve eq.(4.12) we divide the kinematical regions into three cases: (I) $p_1 > k_1,\ k_2 > n-1$, (II) $n-1 > k_1,\ k_2 > p_1$, (III) $k_1 > p_1,\ n-1 > k_2$ or $k_2 > p_1,\ n-1 > k_1$. In case (I) the non-linear terms (the fifth term) vanish. In case (II) the fourth term vanishes due to the Seiberg condition, where note that $k_1 - n + 1 < 0$ and $k_2 - n + 1 < 0$. In case (III) one side in the brackets of the fourth term vanishes and also one of the non-linear terms does. The solution is finally summarized in the form

$$\begin{aligned} \hat{S}^{(0)}_{q_1,q_2\to q_3,q_4} &= \lambda^{-1}\delta(q_1+q_2-q_3-q_4)\prod_{j=1}^{4} q_j \left(q_{max}-1\right) x^{q_1+q_2-2} \\ &= \lambda^{-1}\delta(q_1+q_2-q_3-q_4)\prod_{j=1}^{4} q_j \\ &\qquad \times\frac{1}{2}\left(|q_1+q_2|+|q_1-q_3|+|q_1-q_4|-2\right) x^{q_1+q_2-2}\ , \end{aligned} \tag{4.13}$$

where $q_{max} = max(q_j)$.

Solving the $M = 2$ equation of (4.11) we get the $2 \to 3$ amplitude for all kinematical regions,

$$\hat{S}^{(0)}_{q_1,q_2\to q_3,q_4,q_5} = \lambda^{-1}\delta(q_1+q_2-q_3-q_4-q_5)\prod_{j=1}^{5} q_j\ f(q)\ x^{q_1+q_2-3} \tag{4.14}$$

where

$$f(q) = \left\{ \begin{array}{lcc} (q_{max}-1)(q_{max}-2) & \text{for} & (q_{max}, q_{min}) = (q_1,q_2) \text{ or } (q_2,q_1) \\ (q_{max}-1)(q_{(2)}+q_{(3)}-2) & & \text{otherwise} \end{array} \right. \tag{4.15}$$

where $q_{min} = min(q_j)$. $q_{(2)}$ and $q_{(3)}$ are the second and the third largest momenta in $\{q_j\}$.

4.2 Higher genus amplitudes

Let us first consider the Ward identities of $(r, s) = (-2, -1)$,

$$\frac{1}{\pi}\int \bar{\partial} < W_{-2,-1}\hat{T}^{+}_{k_1}\prod_{j=1}^{M}\hat{T}^{-}_{p_j} >_g = 0\ . \tag{4.16}$$

For $M = 1$ we get the equation

$$
\begin{aligned}
& -x < \hat{T}^+_{k_1} \hat{T}^-_{p_1} \hat{T}^-_1 >_g - p_1 < \hat{T}^+_{k_1} \hat{T}^-_{p_1+1} >_g \\
& +2! x k_1 < \hat{T}^+_{k_1-1} \hat{T}^-_{p_1} >_g \\
& -\frac{\lambda}{2} \int_0^1 dl < \hat{T}^-_{1-l} \hat{T}^-_l \hat{T}^-_{p_1} \hat{T}^+_{k_1} >_{g-1} \\
& +\frac{\lambda}{2} 2! k_1 \int_0^{k_1-1} dl < \hat{T}^+_{k_1-1-l} \hat{T}^+_l \hat{T}^-_{p_1} >_{g-1} = 0
\end{aligned}
\tag{4.17}
$$

and for $M = 2$ we obtain

$$
\begin{aligned}
& -x < \hat{T}^+_{k_1} \hat{T}^-_{p_1} \hat{T}^-_{p_2} \hat{T}^-_1 >_g - p_1 < \hat{T}^+_{k_1} \hat{T}^-_{p_1+1} \hat{T}^-_{p_2} >_g \\
& -p_2 < \hat{T}^+_{k_1} \hat{T}^-_{p_1} \hat{T}^-_{p_2+1} >_g + 2! x k_1 < \hat{T}^+_{k_1-1} \hat{T}^-_{p_1} \hat{T}^-_{p_2} >_g \\
& -\frac{\lambda}{2} \int_0^1 dl < \hat{T}^-_{1-l} \hat{T}^-_l \hat{T}^-_{p_1} \hat{T}^-_{p_2} \hat{T}^+_{k_1} >_{g-1} \\
& +\frac{\lambda}{2} 2! k_1 \int_0^{k_1-1} dl < \hat{T}^+_{k_1-1-l} \hat{T}^+_l \hat{T}^-_{p_1} \hat{T}^-_{p_2} >_{g-1} \\
& +\lambda 2! k_1 \sum_{h=0}^{g} \int_0^{k_1-1} dl < \hat{T}^-_{p_1} \hat{T}^+_{k_1-1-l} >_h < \hat{T}^+_l \hat{T}^-_{p_2} >_{g-h} = 0 \ .
\end{aligned}
\tag{4.18}
$$

We can easily check that these equations satisfy the following higher genus amplitudes. For the $1 \to N-1$ amplitudes at genus one and two they are given by

$$
\hat{S}^{(1)}_{q_1 \to q_2, \cdots, q_N} = \frac{1}{24} \delta\Big(q_1 - \sum_{j=2}^{N} q_j\Big) \prod_{j=1}^{N} q_j \prod_{t=1}^{N-1} (q_1 - t) \Big(\sum_{j=2}^{N} q_j^2 - q_1 - 1\Big) x^{q_1 - N} \tag{4.19}
$$

and

$$
\begin{aligned}
\hat{S}^{(2)}_{q_1 \to q_2, \cdots, q_N} = {} & \frac{\lambda}{5760} \delta\Big(q_1 - \sum_{j=2}^{N} q_j\Big) \prod_{j=1}^{N} q_j \prod_{t=1}^{N+1} (q_1 - t) \ x^{q_1 - N - 2} \\
& \times \Big[3 \sum_{j=2}^{N} q_j^4 + 10 \sum_{\substack{i,j=2 \\ (i<j)}}^{N} q_i^2 q_j^2 - (10 q_1 + 5) \sum_{j=2}^{N} q_j^2 \\
& \qquad + 10 \sum_{\substack{i,j=2 \\ (i<j)}}^{N} q_i q_j + 12 q_1 + 7 \Big]
\end{aligned}
\tag{4.20}
$$

for $N \geq 1$. The $2 \to 2$ amplitudes at genus one is

$$\hat{S}^{(1)}_{q_1,q_2\to q_3,q_4} = \frac{1}{24}\delta(q_1+q_2-q_3-q_4)\prod_{i=1}^{4} q_i \ (q_{max}-1)\ x^{Q-4}$$
$$\times\Big[(Q^3+6Q+1)(Q-2q_{max}-6)+(Q^2+L)(q_{max}^2+13q_{max})$$
$$-4Q(2q_{max}^2-9)-6L(3q_{max}-1)+10q_{max}^2\Big]\ , \quad (4.21)$$

where $Q = q_1 + q_2$ $(= q_3 + q_4)$ and $L = q_{(2)}^2 + q_{(3)}^2$. $q_{(2)}$ and $q_{(3)}$ are the second and the third largest momenta. The first several of (4.19-20) and (4.21) have been derived in the matrix model [1]. Thus we confirm that eqs.(4.17) and (4.18) and also $M > 2$ equations of (4.16) are consistent with the matrix model results.

However, the higher genus amplitudes do not satisfy the general $W_{-n,-m}$ identity with $n, m > 2$. This indicates that further boundary contributions are necessary for the higher genus cases. It is easily imagined that there are the contributions shown in fig.5. On the basis of this figure we can speculate a generalisation of the formula (3.16) as follows:

$$\lambda^{a-1+h}\int\prod_{i=1}^{a} dl_i\theta(l_i)\ D_a^{+(h)}(-l_1,\cdots,-l_a,k_1,\cdots,k_{n+1-a-2h};n,m)$$
$$\times < \hat{T}^+_{l_1}\ \mathcal{O}'_1 >_{g_1} < \hat{T}^+_{l_2}\ \mathcal{O}'_2 >_{g_2}\cdots < \hat{T}^+_{l_a}\ \mathcal{O}'_a >_{g_a}\ , \qquad (4.22)$$

where $\sum_{i=1}^{a} g_i = g - h$ and $a = 1, \cdots, n+1-2h$. h stands for the genus of the surface Σ in fig.5, where the $h = 2$ case is presented. $-l_i$ represents the conjugate mode of $\hat{T}^+_{l_i}$. The $h = 0$ formula is nothing but (3.16). The $h \neq 0$ contributions exist for $g \geq h$ and $n \geq 2h$, where note that the $h = 1$ formula would contibute in the $W_{-2,-1}$ identities, but it vanishes due to the Seiberg condition.

The direct calculation of the D-functions for $h \geq 1$ are very difficult. So we guess the forms. Recall that the discrete state $R_{-n,-m}$ is given by the OPE of the tachyon operators

$$T^-_{n+1}\times T^+_{m+1}\sim R_{-n,-m}\ . \qquad (4.23)$$

This suggests that we could replace the operator $\bar{\partial}W_{-n,-m}$ with the two tachyons $\hat{T}^-_{n+1}$ and $\hat{T}^+_{m+1}$. Thus we identify the surface Σ with the $1 \to$

$n+2-2h$ amplitude of genus h, $\hat{S}^{(h)}_{m+1,-l_1,\cdots,-l_a,k_1,\cdots,k_{n+1-a-2h}\to n+1}$. From this argument we guess the expressions of the D-functions as follows:

$$\begin{aligned} &D^{+(h)}_a(-l_1,\cdots,-l_a,k_1,\cdots,k_{n+1-a-2h};n,m) \\ &\quad = \frac{\lambda^{1-h}}{(n+1)(m+1)\prod_{i=1}^a(-l_i)}\ \hat{S}^{(h)}_{m+1,-l_1,\cdots,-l_a,k_1,\cdots,k_{n+1-a-2h}\to n+1}\ , \end{aligned} \tag{4.24}$$

where the $\hat{S}$-matrix formula is applied as if $-l_i$ were positive. The normalization is fixed by fitting the $h=0$ formula with (3.16). Using the explicit expressions of the amplitudes (4.10), (4.19) and (4.20), we obtain the first few of the D-functions

$$D^{+(0)}_a = \delta(\sum_{i=1}^a l_i - \sum_{i=1}^{n+1-a} k_i + n - m)\ n!\ \Big(\prod_{i=1}^{n+1-a} k_i\Big)\ , \tag{4.25}$$

$$\begin{aligned} D^{+(1)}_a &= \delta(\sum_{i=1}^a l_i - \sum_{i=1}^{n-1-a} k_i + n - m)\ \frac{1}{24}\ n!\ \Big(\prod_{i=1}^{n-1-a} k_i\Big) \\ &\quad \times\Big(\sum_{i=1}^a l_i^2 + \sum_{i=1}^{n-1-a} k_i^2 + (m+1)^2 - n - 2\Big) \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} D^{+(2)}_a = \delta(\sum_{i=1}^a l_i - \sum_{i=1}^{n-3-a} k_i + n - m)\ \frac{1}{5760}\ n!\ \Big(\prod_{i=1}^{n-3-a} k_i\Big) \\ \times\Bigg[3\Big\{\sum_{i=1}^a l_i^4 + \sum_{i=1}^{n-3-a} k_i^4 + (m+1)^4\Big\} + 10\Big\{\sum_{\substack{i,j=1\\(i<j)}}^a l_i^2 l_j^2 + \sum_{\substack{i,j=1\\(i<j)}}^{n-3-a} k_i^2 k_j^2 \\ + \sum_{i=1}^a \sum_{j=1}^{n-3-a} l_i^2 k_j^2 + (m+1)^2\Big(\sum_{i=1}^a l_i^2 + \sum_{i=1}^{n-3-a} k_i^2\Big)\Big\} \\ -\Big(10(n+1)+5\Big)\Big\{\sum_{i=1}^a l_i^2 + \sum_{i=1}^{n-3-a} k_i^2 + (m+1)^2\Big\} \\ +10\Big\{\sum_{\substack{i,j=1\\(i<j)}}^a l_i l_j + \sum_{\substack{i,j=1\\(i<j)}}^{n-3-a} k_i k_j + \sum_{i=1}^a \sum_{j=1}^{n-3-a} (-l_i) k_j \\ +(m+1)\Big(\sum_{i=1}^a (-l_i) + \sum_{i=1}^{n-3-a} k_i\Big)\Big\} + 12(n+1) + 7\Bigg]\ . \end{aligned} \tag{4.27}$$

Similarly, for the negative chirality, we get

$$-\lambda^{b-1+h}\int\prod_{i=1}^{b}dl_i\theta(l_i)\ D_b^{-(h)}(-l_1,\cdots,-l_b,p_1,\cdots,p_{m+1-b-2h};n,m)$$
$$\times<\hat{T}_{l_1}^{-}\ \mathcal{O}_1'>_{g_1}<\hat{T}_{l_2}^{-}\ \mathcal{O}_2'>_{g_2}\cdots<\hat{T}_{l_b}^{-}\ \mathcal{O}_b'>_{g_b}\ , \tag{4.28}$$

where

$$\begin{aligned} &D_b^{-(h)}(-l_1,\cdots,-l_b,p_1,\cdots,p_{m+1-b-2h};n,m) \\ &=\frac{\lambda^{1-h}}{(n+1)(m+1)\prod_{i=1}^{b}(-l_i)}\ \hat{S}^{(h)}_{m+1\to n+1,-l_1,\cdots,-l_b,p_1,\cdots,p_{m+1-b-2h}}\end{aligned} \tag{4.29}$$

and $\sum_{i=1}^{b}g_i=g-h$ and $b=1,\cdots,m+1-2h$.

If the D-functions are given, we can write out all types of the Ward identities. For instance we get

$$\begin{aligned} 0 &= \frac{1}{\pi}\int d^2z\bar{\partial}<W_{-n,-1}\hat{T}_{k_1}^{+}\hat{T}_{p_1}^{-}>_g \\ &= -p_1<\hat{T}_{k_1}^{+}\hat{T}_{p_1+n-1}^{-}>_g-x<\hat{T}_{k_1}^{+}\hat{T}_{p_1}^{-}\hat{T}_{n-1}^{-}>_g \\ &\quad-\frac{\lambda}{2}\int_0^{n-1}dl<\hat{T}_{n-1-l}^{-}\hat{T}_{l}^{-}\hat{T}_{p_1}^{-}\hat{T}_{k_1}^{+}>_{g-1} \\ &\quad+\sum_{a=1}^{g+1}\sum_{h=0}^{g-a+1}\frac{\lambda^{a-1+h}}{a!}\frac{(x/\epsilon)^{n-2h-a}}{(n-2h-a)!}\int\prod_{i=1}^{a}dl_i\theta(l_i) \\ &\qquad\times D_a^{+(h)}(-l_1,\cdots,-l_a,k_1,\epsilon,\cdots,\epsilon;n,1)<\hat{T}_{p_1}^{-}\ \prod_{i=1}^{a}\hat{T}_{l_i}^{+}>_{g-h-a+1}\ .\end{aligned} \tag{4.30}$$

The D-functions are now given up to $h=2$ so that we can calculate all amplitudes up to genus two. In the following we give the procedure to derive the amplitudes more than genus 2. To calculate the three genus amplitudes, let us consider the Ward identities of type

$$\frac{1}{\pi}\int d^2z\bar{\partial}<W_{-3,-2}\hat{T}_{k_1}^{+}\prod_{j=1}^{M}\hat{T}_{p_j}^{-}>_g=0\ . \tag{4.31}$$

For $M=1$ this identity gives

$$-x^2<\hat{T}_{k_1}^{+}\hat{T}_{p_1}^{-}\hat{T}_1^{-}>_g-2xp_1<\hat{T}_{k_1}^{+}\hat{T}_{p_1+1}^{-}>_g$$

$$
\begin{aligned}
&-\lambda x \int_0^1 dl < \hat{T}^-_{1-l}\hat{T}^-_l\hat{T}^-_{p_1}\hat{T}^+_{k_1} >_{g-1} -\lambda p_1 \int_0^{p_1+1} dl < \hat{T}^-_{p_1+1-l}\hat{T}^-_l\hat{T}^+_{k_1} >_{g-1} \\
&-\frac{\lambda^2}{3}\int_0^1 dl \int_0^{1-l} dl' < \hat{T}^-_{1-l-l'}\hat{T}^-_l\hat{T}^-_{l'}\hat{T}^-_{p_1}\hat{T}^+_{k_1} >_{g-2} -\frac{13}{12}\lambda < \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_1 >_{g-1} \\
&+3x^2 k_1 < \hat{T}^+_{k_1-1}\hat{T}^-_{p_1} >_g +3\lambda x k_1 \int_0^{k_1-1} dl < \hat{T}^+_{k_1-1-l}\hat{T}^+_l\hat{T}^-_{p_1} >_{g-1} \\
&+\lambda^2 k_1 \int_0^{k_1-1} dl \int_0^{k_1-1-l} dl' < \hat{T}^+_{k_1-1-l-l'}\hat{T}^+_l\hat{T}^+_{l'}\hat{T}^-_{p_1} >_{g-2} \\
&+\frac{\lambda}{4}k_1[(k_1-1)^2+k_1^2+4] < \hat{T}^+_{k_1-1}\hat{T}^-_{p_1} >_{g-1} = 0 \ , \qquad (4.32)
\end{aligned}
$$

where the 6-th and the last terms of l.h.s. are the contributions from the $D^{-(h=1)}_{a=1}$ and the $D^{+(h=1)}_{a=1}$ formulas respectively.

Combining the identities (4.17) and (4.32) of $g = 3$ and removing the amplitude $< \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_1 >_3$, we get the identity in which the $1 \to 1$ amplitude of genus three is expressed by the lower genus amplitudes. Substituting the explicit expressions of the lower genus amplitudes in the identity we get

$$
\begin{aligned}
&x p_1 < \hat{T}^+_{k_1}\hat{T}^-_{p_1+1} >_3 -x^2 k_1 < \hat{T}^+_{k_1-1}\hat{T}^-_{p_1} >_3 \qquad (4.33) \\
&= \frac{\lambda^2}{483840}\delta(k_1-p_1-1)\ (p_1+1)p_1^2(p_1-1)(p_1-2)(p_1-3)(p_1-4) \\
&\quad \times\Big[18p_1^6 - 84p_1^5 - 35p_1^4 + 238p_1^3 + 161p_1^2 - 112p_1 - 93\Big]\ x^{p_1-4} \ .
\end{aligned}
$$

Solving this equation we get the $1 \to 1$ amplitude of genus three

$$
\begin{aligned}
\hat{S}^{(3)}_{q_1\to q_2} = &\frac{\lambda^2}{2903040}\delta(q_1-q_2)q_1^2\prod_{t=1}^{5}(q_1-t)\ x^{q_1-6} \\
&\times\Big[9q_1^6 - 63q_1^5 + 42q_1^4 + 217q_1^3 - 205q_1 - 93\Big] \ . \qquad (4.34)
\end{aligned}
$$

This agrees with the result computed in [18].

The $1 \to 2$ amplitude of genus 3 is also given by considering the simultaneous equation of the $M = 2$ identities of (4.16) and (4.31). Removing the amplitude $< \hat{T}^+_{k_1}\hat{T}^-_{p_1}\hat{T}^-_{p_2}\hat{T}^-_1 >_3$, the $1 \to 2$ amplitude of $g = 3$ is expressed by the lower genus amplitudes and $1 \to 1$ amplitudes of $g = 3$ obtained in eq.(4.34). Recurrently we can obtain the $1 \to N-1$ amplitude of genus three. Furthermore, using this, we can determine the D-functions of $h = 3$. Then we can get all types of genus three amplitudes. Repeating the calculation mentioned above we can derive amplitudes of $g > 3$.

We finally discuss the partition function of 2D string theory. The second derivative of it with respect to the tachyon background (cosmological constant) x $(=-\mu)$ is given by

$$\frac{\partial^2 Z_g}{\partial x^2}=\left(\frac{\mu_B}{\mu}\right)^2 < T_0\ T_0 >_g = \frac{1}{\epsilon^2} < \hat{T}^+_\epsilon \hat{T}^-_\epsilon >_g \ . \tag{4.35}$$

Therefore, using the resuts of the $1 \to 1$ amplitudes of each genus, we get

$$\frac{\partial^2 Z}{\partial x^2}=\lambda^{-1}\log x+\frac{1}{24}x^{-2}-\lambda\frac{7}{960}x^{-4}+\lambda^2\frac{31}{8064}x^{-6}+\cdots\ , \tag{4.36}$$

where $Z=\sum_g Z_g$. The first term of r.h.s. is given by the algebra $\frac{1}{\epsilon}x^\epsilon = \frac{1}{\epsilon}+\log x$, where the divergent part should be regularized properly. From the second term we get $Z_1=-\frac{1}{24}\log x$. Note that, contrary to the $c_M<1$ case, there is no problem with doubling of the partition functions when we identify the matrix model result and the continuum one because in 2D string theory we consider both chiralities, while in $c_M<1$ only one half of the chiralities are considered.

5 On Differences Between $c_M=1$ and $c_M<1$

The scaling operator of quantum gravity coupled to (p,q) minimal matter, which is subject to the W_q algebra constraints [19], is

$$\hat{O}_j=\frac{\Gamma(j/q)}{\Gamma(-j/q)}O_j\ , \tag{5.1}$$

where $j=1,2,3,\cdots,\ (j\neq q \bmod q)$. O_j is the tachyon operator with the discrete momentum defined in ref. [4]. The operator $\hat{O}_{nq},\ n\in \mathbf{Z}_{>0}$ decouples from the theory since the Γ-factor in (5.1) vanishes at $j=nq$. Recall that, as discussed in Sect.3, in the limit $q\to\infty$ ($c_M\to 1$) and $j/q\to k$, the scaling operator (5.1) becomes the tachyon operator with positive chirality $\hat{T}^+_k$.

We now clarify the structural differences between the $c_M<1$ model satisfying the W-algebra constraints and 2D string theory. In 2D string theory we consider both chiralities of the tachyon operators to define the S-matrix, while in the $c_M<1$ model we consider only one half of the chiralities, where

it is the positive one.[6] Also in $c_M < 1$ theory the operator $\hat{O}_{nq}$ decouples from the theory, but the corresponding tachyon operator $\hat{T}_n^+$ of the $c_M = 1$ model no longer decouples. The factor $\Lambda(k)$ (2.3) vanishes at the momentum $k = n$, but extra divergence appears at $q \to \infty$ so that the $\hat{S}$-matrix becomes finite. It means that the S-matrix of 2D string theory has a pole at the momentum $k = n$ due to the external leg factor $\Lambda^{-1}(k)$.

The pole of the Λ^{-1}-factor plays an important role when we consider spacetime physics [9, 20], which is related to the discrete state through the OPE (3.1). For example, $T_2^+ \times T_2^- \sim \partial X \bar{\partial} X$, where the r.h.s. is the operator giving the deformation of geometry. In fact in ref. [9] the contribution from the $k = 2$ pole gives the gravitational scattering. On the other hand, for $c_M < 1$, the decoupling of the $j = nq$ operators indicates that the discrete states are excluded from the theory.

The extra boundary contributions discussed in Sect.4.2 also seem to appear in the expressions of the W-algebra constraints for $c_M < 1$. For the positive chirality theory the Ward identity of the current $W_{-s+1,-l-s+1}$ is identified with the $\mathcal{W}_l^{(s)}$ constraint [4], where $s = 2$ is the Virasoro constraint. The extra boundary contributions appear only for $s \geq 4$ constraints, independently of l, because the extra boundary for negative chirality (4.28) does not contribute in this case [7].

I am grateful to B. Bullock for careful reading of the manuscript.

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Figure Captions

- Fig.1 (a) The boundary corresponding to the OPE (3.7). The incoming and the outgoing arrows stand for the tachyon vertex operators with $+$ and$-$ chiralities respectively. The W point is the $W_{-n,-m}$ current. After integrating over the locations of the vertex operators $z_2, \cdots, z_n$, we find the $1/z$ pole. (b) The derivative $\bar{\partial}$ picks up the singularity and then we get the 1-b configuration. Here $K = k_1 + \cdots + k_n - n + m$.

- Fig.2 (a) The boundary producing the non-linear term. The cross point stands for the operator $\bar{\partial} W_{-n,-m}$. (b) At $\tau \to \infty$ we get the product of two surfaces. Here $K = k_1 + \cdots + k_{n-1} - n + m$ and l is integrated from 0 to K.

- Fig.3 This is a variant of the boundary of fig.2, where the surfaces Σ_1 and Σ_2 are connected by a handle.

- Fig.4 (a) The boundary for the general formula (3.16). (b) The configuration at $\tau \to \infty$.

- Fig.5 The extra boundary contribution for the case of $h = 2$. The surface Σ has two handles.

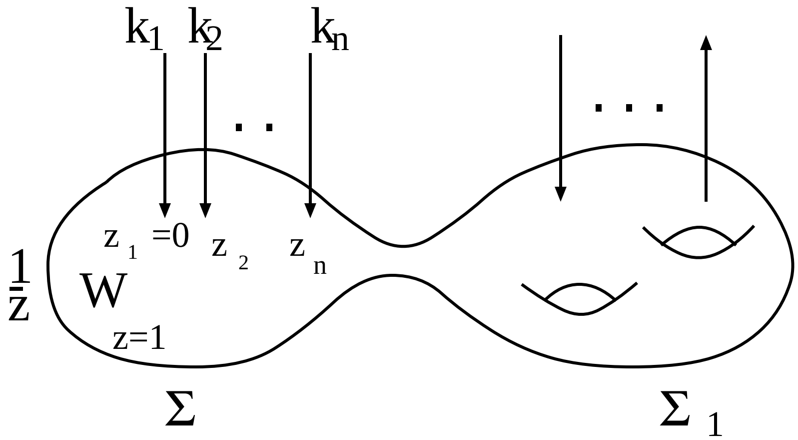


Fig. 1-a

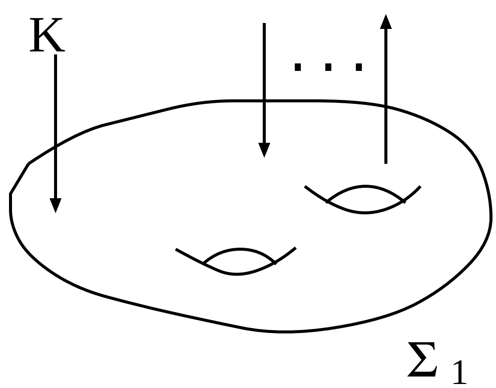


Fig. 1-b

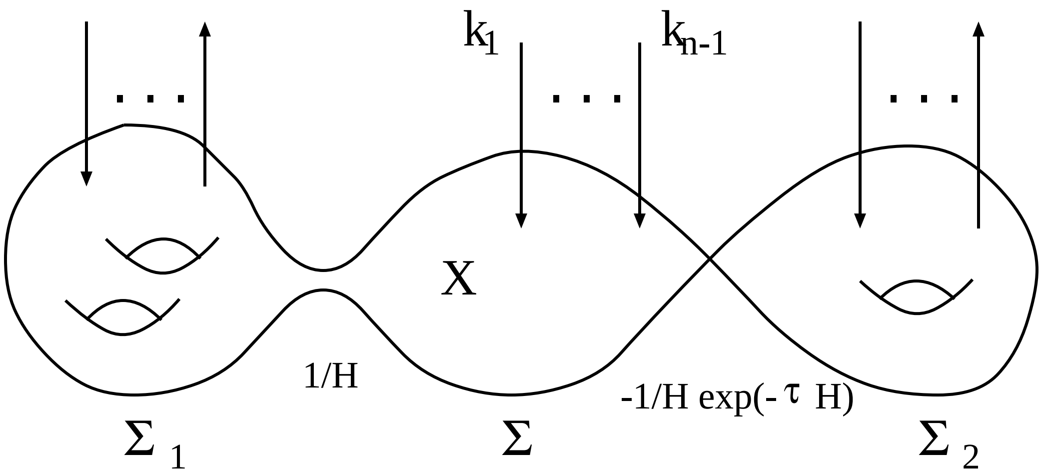
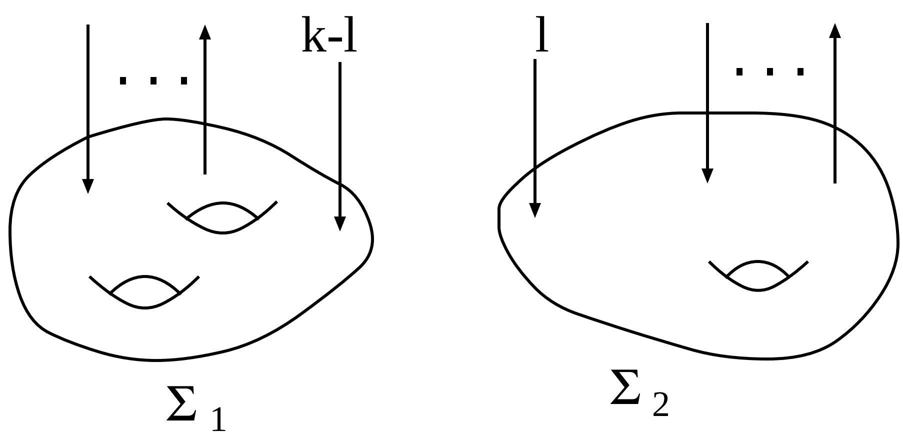


Fig. 2-a



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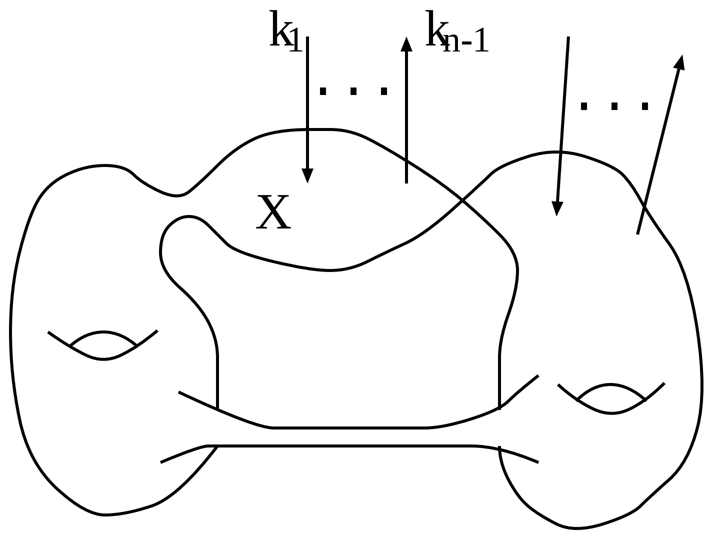

Fig. 3

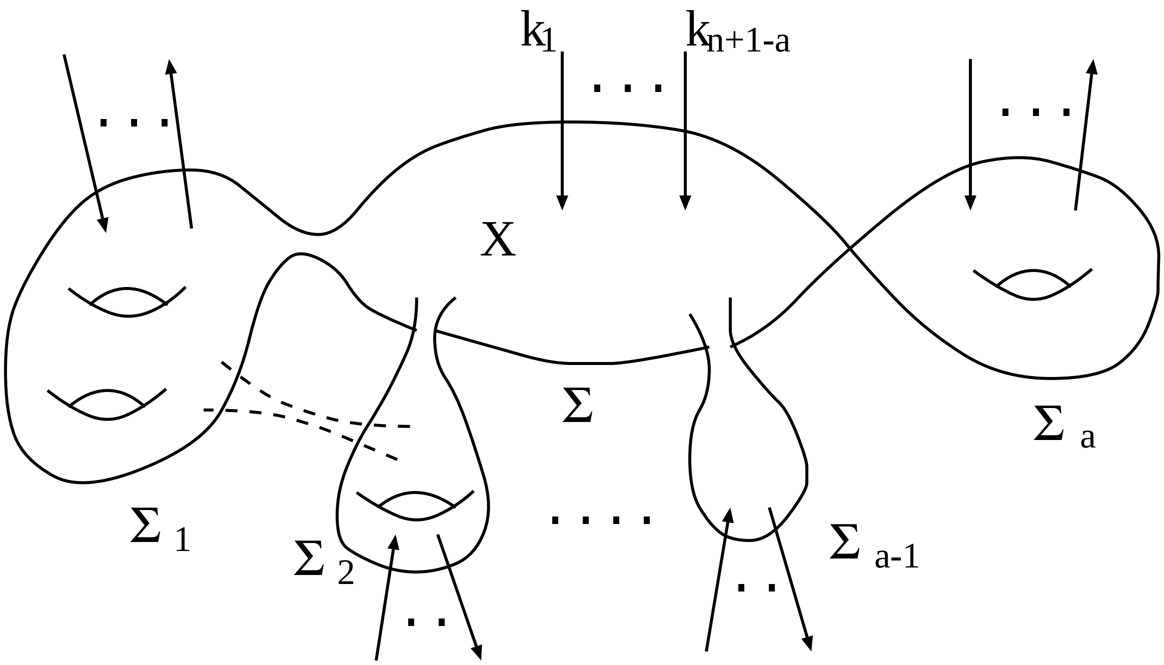

Fig. 4-a

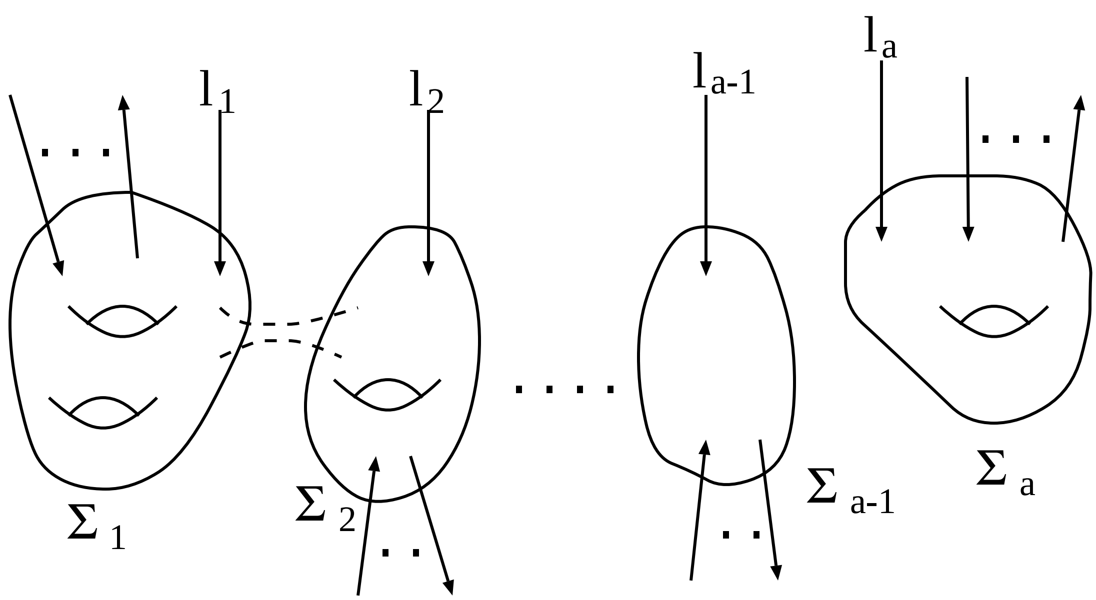

Fig. 4-b

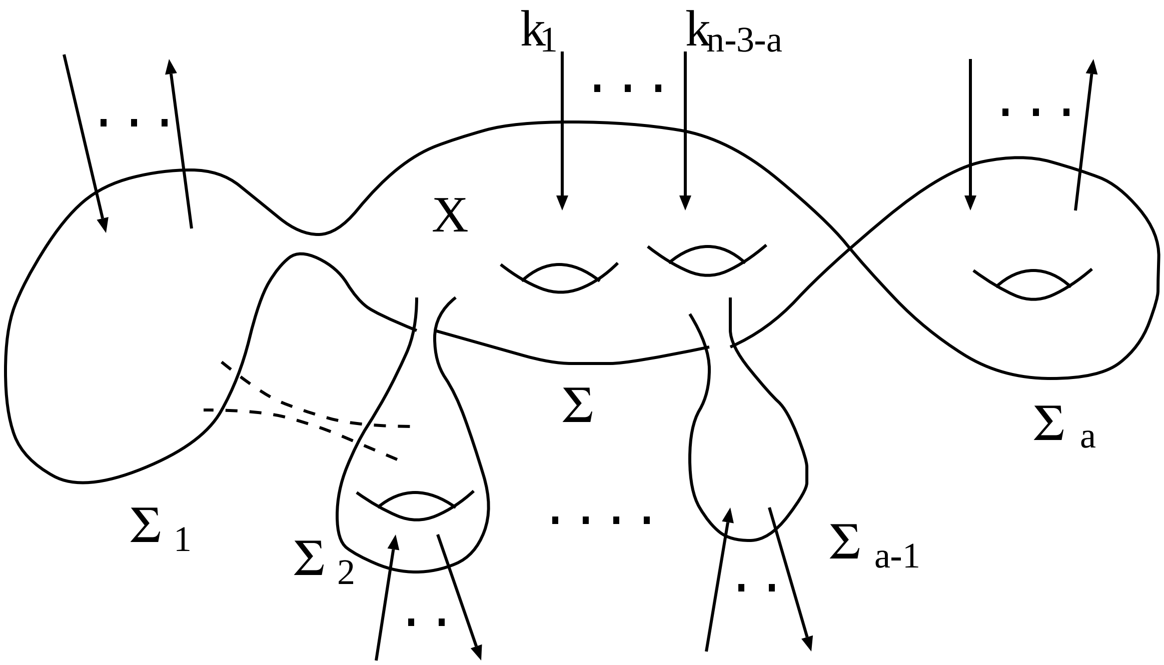

Fig. 5